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$\mathfrak{Z}$-PERMUTABLE SUBGROUPS AND p-NILPOTENCY OF FINITE GROUPS II

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ABSTRACT

Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a finite group G, that is, for each prime p dividing the order of G, $\mathfrak{Z}$ contains one and only one Sylow p-subgroup of G. A subgroup H of G is said to be $\mathfrak{Z}$-permutable in G if H permutes with every member of $\mathfrak{Z}$. In this paper we characterize p-nilpotency of finite groups G; we will assume that some minimal subgroups or 2-minimal subgroups of G are $\mathfrak{Z}$-permutable in G. Moreover, the duals of some recent results are obtained.

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1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [11]. Throughout this article, G stands for a finite group.

A subgroup of G is called **quasinormal** in G if it permutes with every subgroup of G. We say, following Kegel [13], that a subgroup of G is **π-quasinormal** in G if it permutes with every Sylow subgroup of G. Recently, Asaad and Heliel [2] introduced a new embedding property, namely, the $\mathfrak{Z}$-permutability of subgroups of a group. A set $\mathfrak{Z}$ is called *a complete set of Sylow subgroups* of G if for each prime $p \in \pi(G)$ (the set of distinct primes dividing $|G|$), $\mathfrak{Z}$ contains exactly one Sylow p-subgroup of G, say, G_p. A subgroup of G is said to be **$\mathfrak{Z}$-permutable** in G if it permutes with every member of $\mathfrak{Z}$. Obviously, every π-quasinormal subgroup is $\mathfrak{Z}$-permutable. In contrast to the fact that every π-quasinormal subgroup is subnormal (see [13]), it does not hold, in general, that every $\mathfrak{Z}$-permutable subgroup of G is subnormal in G. It suffices to consider the alternating group of degree 4.

Many authors have investigated the structure of a group G under the assumption that some maximal subgroups of some subgroups of G are well-situated in G. Srinivasan [19] proved that a group G is supersolvable if every maximal subgroup of any Sylow subgroup of G is normal. Later on, Wall [20] gave a complete classification of finite groups under the assumption of Srinivasan. From [1, Theorem 3.1], we know that if G is a group and p the smallest prime dividing $|G|$, then G is p-nilpotent if the maximal subgroups of the Sylow p-subgroups of G are π-quasinormal in G. Moreover, Asaad and Heliel proved in [2] that if $\mathfrak{Z}$ is a complete set of Sylow subgroups of a group G and if the maximal subgroups of G_p are $\mathfrak{Z}$-permutable subgroups of G, for all $G_p \in \mathfrak{Z}$, then G is p_1-nilpotent, where p_1 is the smallest prime dividing $|G|$. In [15], the authors generalize the above mentioned results by obtaining the following results [15, Theorem 3.1 and 3.4]: Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G and p the smallest prime dividing $|G|$. Then G is p-nilpotent if one of followings holds: (1) the maximal subgroups of $G_p \in \mathfrak{Z}$ are $\mathfrak{Z}$-permutable subgroups of G; (2) G is A_4-free and the 2-maximal subgroups of G_p are $\mathfrak{Z}$-permutable subgroups of G.

The minimal subgroups of G are the subgroups of G of prime order. **A 2-minimal subgroup** of G is the subgroup which contains a minimal subgroup of G as its maximal subgroup. Obviously, the subgroups of G of prime square order are a kind of 2-minimal subgroups of G.

As we know, the concepts of maximal subgroup and minimal subgroup are dual in the finite group theory, so influences of the properties of the minimal subgroups of a finite group G on the structure of G were investigated by many authors. For example, Itô has proved that if the center of a group G of odd order contains all minimal subgroups of G, then G is nilpotent. An extension of Itô's result is the following statement [11, p. 435, Satz 5.5]: if, for an odd prime p, every subgroup of G of order p lies in the center of G, then G is p-nilpotent; if all elements of G of order 2 and 4 lie in the center of G, then G is 2-nilpotent. In this paper we want to go further in this direction. We receive the following results, which are the duals of results in [15]:

Theorem 3.1: *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Suppose p is the smallest prime dividing the order of G and P is the Sylow p-subgroup in $\mathfrak{Z}$. If every cyclic subgroup of P of prime order or order 4 (when $p = 2$) is $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.*

Theorem 3.4: *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Suppose that p is the smallest prime dividing the order of G, G is A_4-free and P is the Sylow p-subgroup in $\mathfrak{Z}$. If every subgroup of P of prime square order is $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.*

Let $\mathfrak{F}$ be a class of groups. We call **$\mathfrak{F}$ a formation** provided that (i) if $G \in \mathfrak{F}$ and $H \lhd G$, then $G/H \in \mathfrak{F}$, and (ii) for all normal subgroups M and N of G, if G/M and G/N are in $\mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be **saturated** if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. In this paper, $\mathcal{U}$ will denote the class of supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation ([11, p. 713, Satz 8.6]).

Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. If $N \lhd G$, we denote

$$\mathfrak{Z}N = \{G_pN : G_p \in \mathfrak{Z}\},$$
$$\mathfrak{Z}N/N = \{G_pN/N : G_p \in \mathfrak{Z}\},$$
$$\mathfrak{Z} \cap N = \{G_p \cap N : G_p \in \mathfrak{Z}\}.$$

Suppose P is a p-group, we denote $\Omega(P) = \Omega_1(P)$, if $p > 2$ and $\Omega(P) = \Omega_2(P)$, if $p = 2$.

2. Preliminaries

The following lemmas will be used in the proofs of our results.

Lemma 2.1 ([2, Lemma 2.1]): *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G, U a $\mathfrak{Z}$-permutable subgroup of G, and N a normal subgroup of G. Then:*

(1) *$\mathfrak{Z} \cap N$ and $\mathfrak{Z}N/N$ are complete sets of Sylow subgroups of N and G/N, respectively.*
(2) *UN/N is a $\mathfrak{Z}N/N$-permutable subgroup of G/N.*
(3) *If $U \leq N$, then U is a $\mathfrak{Z} \cap N$-permutable subgroup of N.*

Lemma 2.2 ([10, Lemma 3.1]): *Let P be a normal p-subgroup of a group G and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. If every cyclic subgroup of P of order p or order 4 (if $p = 2$) is $\mathfrak{Z}$-permutable in G, then every cyclic subgroup of P of order p or order 4 (if $p = 2$) is π-quasinormal in G.*

Lemma 2.3 ([12, X 13]): *Let G be a group and M a subgroup of G.*

(a) *If M is normal in G, then $F^*(M) \leq F^*(G)$, where $F^*(G)$ is the generalized Fitting subgroup of G;*
(b) *If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

Lemma 2.4 ([16, Theorem 3.1 and 3.3]): *Let $\mathfrak{F}$ be a saturated formation containing $\mathcal{U}$, and let G be a group. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and every cyclic subgroup of $F^*(H)$ of prime order or order 4 is π-quasinormal in G.*

Lemma 2.5 ([21, Lemma 3.12]): *Let p be the smallest prime dividing the order of G. If the order of G is not divisible by p^3 and G is A_4-free, then G is p-nilpotent.*

Lemma 2.6 ([15, Lemma 2.6]): *Suppose that G is a finite nonabelian simple group. Then there exists an odd prime $r \in \pi(G)$ such that G has no $\{2, r\}$-Hall subgroup.*

3. Main Results

Theorem 3.1: *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of a group G. Suppose p is the smallest prime dividing the order of G and P*

is the Sylow p-subgroup in $\mathfrak{Z}$. If every cyclic subgroup of P of prime order or order 4 (when $p = 2$) is $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.

Proof. Assume that the result is false, and let G be a counterexample of minimal order. Then we have:

(1) $O_{p'}(G) = 1$.
If $O_{p'}(G) \neq 1$, consider the quotient group $G/O_{p'}(G)$. For any cyclic subgroup $\overline{U}$ of prime order or order 4 (when $p = 2$) of $PO_{p'}(G)/O_{p'}(G)$, we can write $\overline{U} = UO_{p'}(G)/O_{p'}(G)$, where U is a cyclic subgroup of prime order or order 4 (when $p = 2$) of P. By hypotheses U is $\mathfrak{Z}$-permutable in G, thus $\overline{U}$ is $\mathfrak{Z}O_{p'}(G)/O_{p'}(G)$-permutable in $GO_{p'}(G)/O_{p'}(G)$ by Lemma 2.1(2). Hence we get that the quotient group $G/O_{p'}(G)$ satisfies the hypotheses of the theorem, so $G/O_{p'}(G)$ is p-nilpotent by the minimal choice of G. Therefore, G is p-nilpotent, a contradiction.

(2) If G is not simple, $F(G) = O_p(G)$ is the maximal normal subgroup of G. Therefore, $G/F(G)$ is a simple group.
Suppose N is a proper normal subgroup of G. By Lemma 2.1(3), every cyclic subgroup of prime order or order 4 (when $p = 2$) of $P \cap N$ is $\mathfrak{Z} \cap N$-permutable in N, hence N satisfies the hypotheses of the theorem. Therefore N is p-nilpotent by the minimal choice of G. By (1) N is a p-group. So $F(G) = O_p(G)$ is the maximal normal subgroup of G.

Notice that there is some prime $q \neq p$ in $\pi(G)$, for otherwise G would be a p-group.

(3) For any Sylow q-subgroup $Q \in \mathfrak{Z}$, where $q \neq p$, $O_p(G)Q = O_p(G) \times Q$.
For any cyclic subgroup $\langle x \rangle$ of prime order or order 4 (when p = 2) of $O_p(G)$, $\langle x \rangle Q$ is a subgroup of G by the hypotheses. Hence, $\langle x \rangle$ normalizes Q by [11, IX, Satz 2.8]. Thus, $[\langle x \rangle, Q] \leq Q \cap O_p(G) = 1$, hence, $\langle x \rangle Q = \langle x \rangle \times Q$, it follows that Q centralizes $\Omega(O_p(G))$. It follows that Q centralizes $O_p(G)$ by [11, IX, Satz 5.12].

(4) $F(G) = Z(G)$, $p = 2$ and G is a quasisimple group.
By (3) we know that $Q \leq C_G(O_p(G))$ for any Sylow q-subgroup $Q \in \mathfrak{Z}$, where $q \neq p$. It follows that $Q^g \leq C_G(O_p(G))$, $\forall g \in G$. By (2) we get that $G = O^p(G) \leq C_G(O_p(G))$. Hence, $F(G) = O_p(G) = Z(G)$. It is easy to see that G is nonsolvable, so $p = 2$ by the famous Odd Order Paper. Hence G is a quasisimple group by (2) and the minimality of G.

(5) The final contradiction. Now we use the Classification of Finite Simple Groups to develop the final contradiction.

Let $\overline{G} = G/Z(G)$. By (4), G is a quasisimple group. By [8, pp. 43–44] $Z(G)$ is a subgroup of the Schur multiplier of $\overline{G}$. From (1) and (4), we have that $p = 2$ and $Z(G)$ is 2-group. By [8, Table 4.1, pp. 302–303], $Z(G)$ is cyclic or $\exp(Z(G)) \leq 4$.

Let $K_1 = \Omega_1(P)$, $K_2 = \Omega_2(P)$ and let $H_1 = \Omega_1(\overline{P})$. For any Sylow r-subgroup Q in $\mathfrak{Z}$, by the hypotheses, $\langle x\rangle Q = Q\langle x\rangle$ for any $x \in K_i$, which implies that $QK_i = K_iQ$ and K_i normalizes Q for $i = 1, 2$. If $K_i = P$, then $\overline{G}$ has a $\{2, r\}$-Hall subgroups, which contradicts Lemma 2.6. Hence, we may assume that $K_i \neq P$ and $\overline{K_i} \neq \overline{P}$ for $i = 1, 2$. If all elements of orders 2 and 4 in G are in $Z(G)$, by [11, pp. 435, Satz 5.5], G is 2-nilpotent. Hence, we may suppose that $\overline{K_2} \neq \overline{1}$.

(5.1) $\overline{G}$ is not a simple group of Lie type in characteristic s over $GF(q)$, where $q = s^f$ and s is prime.

Suppose that $\overline{G}$ is a simple group of Lie type over $GF(q)$, where $q = s^f$ and s is a prime.

(i) Case $s = 2$s: Suppose $\overline{G} \cong A_2(2^2)$. Let Q be the Sylow 7-subgroup of G in $\mathfrak{Z}$. Then $\overline{K_2}$ normalizes $\overline{Q}$. By [5, p. 23], $\overline{K_2Q}$ must be contained in a maximal subgroup $M \cong L_2(7)$ of $A_2(2^2)$ by conjugation. Then by [5, p. 3], $|N_M(M_7)| = 21$, contrary to the fact $\overline{K_2Q} \leq N_M(M_7)$.

Suppose $\overline{G} \neq A_2(2^2)$. Since $Z(G)$ is a 2-group and $(q\pm1, |Z(G)|) = 1$, by [8, Table 4.1, pp. 302–303], $Z(G)$ is isomorphic to a subgroup of $Z_2 \times Z_2$. For $x \in G$ and $\overline{x} \in H_1$, since $x^2 \in Z(G)$, we have that $xZ(G) \subseteq K_2$, hence $H_1 \subseteq \overline{K_2}$. By [3, Theorem 12.1.1, p. 190; Theorem 13.6.4, p. 235], we have $\overline{P} = \overline{K_2}$, a contradiction.

(ii) Case $s > 2$: Suppose that $\overline{G} \cong U_4(3)$, then $|Z(G)|$ divides 4 and $Z(G)$ is a cyclic group. The argument similar to that in the proof of the case $\overline{G} \cong A_2(2^2)$ gives a contradiction.

Suppose that $\overline{G} \cong A_l(q)$, $D_l(q)$, ${}^2D_l(q)$ and ${}^2A_l(q)$. It follows from the structure of $\overline{G}_2$ given in [4] and [8, Theorem 2.15] that there exist an element v in K_1 or K_2 such that $(N_{\overline{G}}(\overline{G}_s))^{\overline{v}} \neq N_{\overline{G}}(\overline{G}_s)$. On the other hand, by the hypotheses, we have $(\overline{G}_s)^{\overline{v}} = \overline{G}_s$ and $(N_{\overline{G}}(\overline{G}_s))^{\overline{v}} = N_{\overline{G}}(\overline{G}_s)$, a contradiction.

Suppose that $\overline{G}$ is not isomorphic to one of $U_4(3)$, $A_l(q)$, $D_l(q)$, ${}^2D_l(q)$ and ${}^2A_l(q)$. Then by [8, Table 4.1, pp. 302–303], $Z(G)$ is a subgroup of Z_2. We have $H_1 \subseteq \overline{K_2}$. Let $x \in K_2$ and $x \notin Z(G)$. Let Q be the Sylow r-subgroup of G in $\mathfrak{Z}$. Then $\langle x\rangle Q = Q\langle x\rangle$, $\langle \overline{x}\rangle\overline{Q} = \overline{Q}\langle\overline{x}\rangle$. Thus $\langle H_1\rangle$ normalizes Q. Taking $r = s$, we get that H_1 is a subgroup of a Cartan subgroup of $\overline{G}$, hence H_1 is an abelian group and . It is easy to see that H_1 is strongly closed in $\overline{P}$ with respect to $\overline{G}$. Since $s > 2$ and $H_1 = \langle H_1\rangle$ by [8, p. 237, Theorem 4.128], $\overline{G}$ is one of $L_2(q)$, $q \equiv 3, 5 (\mathrm{mod}\, 8)$, ${}^2G_2(3^n)$, n odd, $n > 1$. If $\overline{G}$ is $L_2(q)$, $q \equiv 3, 5(\mathrm{mod}\, 8)$, then $|K_2| = 4$ or 8 and $\langle\overline{K_2}\rangle = \overline{P}$, a contradiction. If $\overline{G}$ is ${}^2G_2(3^n)$, then the Schur multiplier of $\overline{G}$ is 1, $\overline{G} = G$, $|K_2| = 8$ and $K_2 = P$ is an elemental abelian group of order 8 by [8, Theorem 3.33], a contradiction.

(5.2) $\overline{G}$ is not an alternating group A_n with $n > 4$.

Suppose that $\overline{G} = A_n$ is an alternating group, where $n > 4$. By Kalužnin's Theorem [18, Theorem 1.6.19], we have $K_2 = P$, a contradiction.

(5.3) $G/Z(G)$ is not a sporadic simple group.

Suppose that $\overline{G}$ is a sporadic simple group. Let $r = \max \pi(\overline{G})$, then $\overline{K_2}$ normalizes $\overline{G}_r$. Suppose $\overline{G} = M_{22}$. Since $\overline{K_2Q}$ is contained in a maximal subgroup in M_{22}, by [5, p. 39], $\overline{K_2Q} \leq L_2(11)$ and $r = 11$. From the maximal subgroups of $L_2(11)$ listed in [5, p. 7], $\overline{K_2} = 1$, a contradiction. Thus $\overline{G} \neq M_{22}$. Then $Z(G) \leq Z_2$ by [8, Table 4.1, pp. 302–303] and $H_1 \subseteq \overline{K_2}$. Since $|H_1| \geq 4$ and $\overline{H_1}$ normalizes $\overline{G}_r$, we have that 4 divides $|N_{\overline{G}}(\overline{G}_r)|$. By [9, pp. 40–69], the only possibility is $\overline{G} \cong He$ and $|N_G(G_{17})| = 17 \cdot 8$. Since the Schur multiplier of He is 1, $G \cong He$. By [5, p. 104], there exists a maximal subgroup $S_4 \times L_3(2)$ in He, hence $|K_2| > 8$. Since K_2 normalizes G_{17}, $|K_2|$ divides $|N_G(G_{17})|$, a contradiction.

Therefore, $G/Z(G)$ is not a simple group, contrary to (4), completing the proof. ∎

Let $p_1 > p_2 > \cdots > p_r$ be the distinct primes dividing $|G|$. Then G is said to satisfy the **Sylow tower property** if there exist $G_{p_1}, G_{p_2}, \ldots, G_{p_r}$ such that G_{p_i} is a Sylow p_i-subgroup of G and $G_{p_1}G_{p_2}\cdots G_{p_k} \lhd G$ for $k = 1, 2, \ldots, r$.

Corollary 3.2: *Let G be a group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. If every subgroup of prime order or order 4 of $G_p \in \mathfrak{Z}$ is $\mathfrak{Z}$-permutable in G, then G satisfies the Sylow tower property.*

Proof. Let p be the smallest prime dividing the order of G and $P \in \mathfrak{Z}$. By Theorem 3.1 G is p-nilpotent. Let N be a normal p-complement of G. Clearly N satisfies the hypotheses of G and, therefore, by induction N satisfies the Sylow tower property. This proves that G satisfies the Sylow tower property. ∎

As an application of Theorem 3.1 we next give a new and shorter proof of the main result of [10].

Corollary 3.3: *Let $\mathfrak{F}$ be a saturated formation containing $\mathcal{U}$, the class of supersolvable groups, and $\mathfrak{Z}$ a complete set of Sylow subgroups of a group G. The following statements are equivalent:*

(i) $G \in \mathfrak{F}$;
(ii) *There is a normal subgroup H of G such that $G/H \in \mathfrak{F}$ and the cyclic subgroups of $G_p \cap F^*(H)$ of prime order or order 4 (if $p = 2$) are $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$, where $F^*(H)$ is the generalized Fitting subgroup of H.*

Proof. By Lemma 2.1 and hypotheses, we know that the cyclic subgroups of $F^*(H)$ of prime order or order 4 are $\mathfrak{Z} \cap F^*(H)$-permutable in $F^*(H)$, thus Corollary 3.2 implies that $F^*(H)$ satisfies the Sylow tower property. In particular, $F^*(H)$ is solvable, hence $F^*(H) = F(H)$ by Lemma 2.3(2). Now hypotheses and Lemma 2.2 imply that the cyclic subgroups of $F^*(H) = F(H)$ of prime order or order 4 is π-quasinormal in G. Applying Lemma 2.4 we get that $G \in \mathfrak{F}$. ∎

Theorem 3.4: *Let G be a finite group and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Suppose p is the smallest prime dividing the order of G, G is A_4-free and P is the Sylow p-subgroup in $\mathfrak{Z}$. If every subgroup of P of prime square order is $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.*

Proof. Assume that the result is false, and let G be a counterexample of minimal order. Then we have:

(1) $O_{p'}(G) = 1$.
If $O_{p'}(G) \neq 1$. It is easy to see that the quotient group $G/O_{p'}(G)$ satisfies the hypotheses of the theorem, hence $G/O_{p'}(G)$ is p-nilpotent by the minimal choice of G. Therefore, G is p-nilpotent, a contradiction.

(2) If G is not simple, $F(G) = O_p(G)$ is the maximal normal subgroup of G. Therefore, $G/F(G)$ is a simple group.
Suppose N is a proper normal subgroup of G. If $|N_p| \leq p^2$, Lemma 2.5 implies that N is p-nilpotent. Using (1), we obtain that $N = N_p$. Hence, assume that $|N_p| > p^2$. By Lemma 2.1, we know that N satisfies the hypotheses of the theorem, hence N is p-nilpotent by the minimal choice of G. By (1) N is a p-group. Hence $F(G) = O_p(G)$ is the maximal normal subgroup of G.

Notice that there is some prime $q \neq p$ in $\pi(G)$, for otherwise G would be a p-group.

(3) For any Sylow q-subgroup $Q \in \mathfrak{Z}$, where $q \neq p$, $O_p(G)Q = O_p(G) \times Q$.
If $|O_p(G)| \leq p^2$, then $O_p(G)Q$ is p-nilpotent by Lemma 2.5. Thus $O_p(G)Q = O_p(G) \times Q$. Suppose $|O_p(G)| > p^2$. For any subgroup X of $O_p(G)$ of prime square order, XQ is a subgroup of G by the hypotheses. Hence X normalizes Q by Lemma 2.5.

On the other hand, $[X, Q] \leq Q \cap O_p(G) = 1$, hence $XQ = X \times Q$. It follows that Q centralizes $\Omega_2(O_p(G))$, in particular, Q centralizes $\Omega(O_p(G))$. Therefore, Q centralizes $O_p(G)$ by [11, IX, Satz, 5.12].

(4) $F(G) = Z(G)$, so G is quasisimple.
By (3) we know that $Q \leq C_G(O_p(G))$, it follows that $Q^g \leq C_G(O_p(G))$, $\forall g \in G$. By (2) we get that $G = O^p(G) \leq C_G(O_p(G))$. Hence $F(G) = O_p(G) = Z(G)$. It is easy to see that G is non-solvable, so $p = 2$ by the famous Odd Order Paper. Hence G is a quasisimple group by (2) and the minimality of G.

(5) The final contradiction.
Now we use the Classification of Finite Simple Groups to develop the final contradiction.

Let $\overline{G} = G/Z(G)$ and $K_2 = \Omega_2(P)$. By (4), G is a quasisimple group. If $K_2 \leq Z(G)$, then G is 2-nilpotent by [11, p. 435, Satz 5.5]. Hence we can assume that $\overline{K}_2 \neq 1$. Let Q be the Sylow r-subgroup of G in $\mathfrak{Z}$ and L be any subgroup of P of order 4. Then $LQ = QL$ by the hypotheses, hence $\overline{LQ} = \overline{QL}$. It follows that $\overline{\langle K_2 \rangle Q} = \overline{Q \langle K_2 \rangle}$ and $\overline{\langle K_2 \rangle Q} \leq \overline{G}$. By [5], it is easy to see that every sporadic simple group has the section A_4. Since $A_4 \leq A_n$, $n \geq 5$, the Alternating groups A_n with $n \geq 5$ have the section A_4. Hence, we may suppose that $\overline{G}$ is a group of Lie type. By Dickson's Theorem, $PSL(2,7) \cong PSL(3,2)$ and $PSL(2,q)$, where $q = p^f$ and p is odd, have the section A_4. Thus, by the information about groups of Lie type in [17, Table 5.1] and [14, Table 3.5 A-3.5 F], if G is A_4-free it is isomorphic to one of the groups: $PSL(2, 2^f)$, f odd, ${}^2B_2(2^f)$, f odd. If $\overline{G}$ is one of $PSL(2, 2^f)$ and ${}^2B_2(2^f)$, where f is odd, the Schur multiplier of $\overline{G}$ is 1 and $\overline{G} = G$. Let r be the largest primitive prime divisor of $2^{2f} - 1$ when $\overline{G}$ is $PSL(2, 2^f)$ and the largest primitive prime divisor of $2^{4f} - 1$ when $\overline{G}$ is ${}^2B_2(2^f)$. By [7], $\overline{G}$ has no the maximal subgroup including $\overline{\langle K_2 \rangle Q}$, the final contradiction. ∎

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